Iterates of a Number-Theoretic Function

By Mohan Lal*

Abstract. Iterates of a function defined by the sum of the prime divisors of a number, where the multiple factors are counted multiply, are considered. The process of iteration is terminated at a prime. The density distribution of these primes is investigated empirically, for $N \leq 60000$ and it is found to be quite constant.

Introduction. Let $n = \prod_{i} p_i^{\alpha_i}$ be the representation of n as a product of distinct primes and define a function,

(1)
$$J(n) = \sum_{i} \alpha_{i} p_{i}.$$

Thus the function J(n) is defined to be the sum of the prime divisors of n and the multiple factors are counted multiply. From (1) it follows that J(n) is completely additive, for J(mn) = J(m) + J(n).

For n, a prime, J(p) = p and for n, a composite number, J(n) < n. For n = 4, J(4) = 4 and this is considered to be an exceptional case and thus the number 4 behaves like a prime.

The *r*th iterate of J(n) is defined by

(2)
$$J_r(n) = J(J_{r-1}(n)); \quad J_1(n) = J(n).$$

When n is a prime, each successive iterate gives rise to the same prime and we may say that the process of iteration converges. In what follows we will assign a value of 1 to r, for n = prime. For composite numbers, r takes definite positive integral values for the iteration (2) to converge. Thus for n = 8, r = 3 since $J_1(8) = 6$; $J_2(8) = 5$; $J_3(8) = 5$. In this manner, we shall associate with an integer n, a function R(n) which defines the minimum number of iterates of J(n) required to transform it into a prime. Thus, we define R(n) = r and so, R(30) = 3, R(24) = 4, R(10) = 2.

Naturally the question arises, "What can be said about R(n)?"

Still, another interesting problem related to the iterates of J(n) could be stated as follows. Suppose we apply the iterates of J(n) to all integers less than or equal to N, we obtain a set of primes which are distributed all over the interval. Then, if $n(p_i)$ is the number of primes p_i between 1 and N inclusive, "Does the ratio $n(p_i)/N$ approach a definite limit, as N approaches infinity, for all p_i ?" It should be stated that the prime p = 3 occurs only once and n = 4 is an exceptional case for which R(4) is not defined in our context.

To find analytic solutions to these questions, which involve various partitions of a number into primes, would probably be difficult. In what follows we provide an empirical investigation of these problems.

Received January 12, 1968, revised July 1, 1968.

^{*} This work was partially supported by the National Research Council under the Grant No. A4026.

TABLE 1 T Distribution of Primes 5 7 11 13 17 19 23 29 31 37 41 43 47 53 1426 810 327 374 152 184 121 62 56 51 70 40 37 5333 2005 649 714 306 533 2341 156 139 97 5544 3096 1556 177 538 236 917 538 235 139 17 19 37 41 43 47 53 5544 3096 1556 177 538 341 156 139 17 139 97 533 170 40 37 533 170 40 37 533 170 40 37 533 233 156 334 156 334 155 333 233 157						
TABLE 1 Distribution of Primes 5 7 11 13 17 19 23 29 31 37 41 43 47 5 7 11 13 17 19 23 29 31 37 41 43 47 5 7 11 13 17 19 23 29 31 37 41 43 47 5534 3205 1541 152 186 533 231 231 341 172 199 37 41 43 47 43 47 43 47 43 47 43 47 43 47 43 47 43 47 47 47 47 47 47 43 47 49 46 49 43 47 49 46 49 47 49 47 49 49 49 49 49 49		59	$\begin{array}{c} 28\\50\\77\\101\\121\\171\\171\\171\\1225\\225\\225\\225\\225\\225\\225\\225\\225\\2$		59	$1.21 \\ 1.20 \\ 1.18$
TABLE 1 TABLE 1 Distribution of Primes 5 7 11 13 17 19 23 29 31 37 41 43 5 7 11 13 17 19 23 29 31 37 41 43 5534 3207 949 714 306 577 213 134 172 104 88 170 1 5534 3202 1241 1400 677 780 426 235 316 200 189 170 1 5534 3202 1241 1400 677 780 426 235 316 200 189 210 104 305 517 104 88 170 1 43 2 143 231 143 231 143 231 143 231 143 235 235 236 236 236 236 236 236 236 <		- 53	$\begin{array}{c} 37\\76\\97\\1128\\156\\187\\221\\253\\312\\3315\\3378\\3378\end{array}$	TABLE 2 Values of $C(p_i)$	53	1.35 1.33 1.33
TABLE 1 Distribution of Primes 5 7 11 13 17 19 23 29 31 37 41 5 7 11 13 17 19 23 29 31 37 41 1426 810 327 374 152 184 121 62 51 39 1 37 41 5534 3202 1241 1400 677 780 426 235 316 200 180 2 396 56 51 189 172 104 877 213 183 21 180 2 237 235 239 239 239 239 239 239 239 236 136 130 135 330 136 130 136 56 51 7 433 331 44 47 336 57 469 7 7 469 7 7 469 <td< td=""><td></td><td>47</td><td>$\begin{array}{c} 40\\ 88\\ 88\\ 139\\ 139\\ 123\\ 223\\ 334\\ 375\\ 340\\ 449\\ 449\\ 492\\ 492\\ \end{array}$</td><td>47</td><td>$1.71 \\ 1.54 \\ 1.48$</td></td<>		47	$\begin{array}{c} 40\\ 88\\ 88\\ 139\\ 139\\ 123\\ 223\\ 334\\ 375\\ 340\\ 449\\ 449\\ 492\\ 492\\ \end{array}$		47	$1.71 \\ 1.54 \\ 1.48 $
TABLE 1 Distribution of Primes 5 7 11 13 17 19 23 29 31 37 5 7 11 13 17 19 23 29 31 37 1426 810 327 374 152 184 121 62 56 56 5330 1605 649 714 306 573 306 183 172 104 5330 1605 656 1737 856 976 538 241 156 5597 2156 2421 1203 1358 788 2404 552 332 9547 5597 2156 1737 856 976 532 332 9547 5597 2156 1233 1358 708 404 552 334 10576 9509 3662 4101 1067 1944 2351 1356 693 927 572 105206 9509 3662 4111 1		43	$\begin{array}{c} 70\\113\\170\\221\\222\\2346\\3346\\3346\\3346\\5520\\5520\\5520\\543\\643\\705\end{array}$		43	$1.79 \\ 1.83 \\ 1.90$
TABLE 1 Distribution of Primes 5 7 11 13 17 19 23 29 31 5 7 11 13 17 19 23 29 31 5534 3207 374 152 184 121 62 86 5534 3202 1241 1400 677 573 306 183 241 5534 3202 1241 1400 677 573 306 183 241 5534 3202 1241 1400 677 780 426 535 316 6547 5597 2156 2421 1203 1358 788 404 552 9547 5597 2156 2421 1203 1356 693 927 10879 6380 2167 1366 1175 520 776 11857 766 1369 2156 1012 252 776	TABLE 1 Distribution of Primes	41	$\begin{array}{c} 51\\ 88\\ 88\\ 139\\ 180\\ 221\\ 221\\ 221\\ 233\\ 331\\ 333\\ 333\\ 336\\ 336\\ 336\\ 336\\ 3$		41	1.37 1.26 1.19
TABLE 1 Distribution of Primes 5 7 11 13 17 19 23 29 31 1426 810 327 374 152 184 121 62 86 2534 3202 1241 1049 497 573 306 183 241 472 5534 3202 1241 1049 497 573 306 183 241 472 5534 3202 1241 1049 497 573 306 183 241 472 5534 3202 1241 1049 497 573 306 183 241 472 9547 559 2156 1737 856 976 332 316 555 776 9547 5302 1356 1057 1356 693 927 703 10879 6380 2351 1356 111 1964 235		37	$\begin{array}{c} 56\\ 104\\ 156\\ 235\\ 235\\ 235\\ 235\\ 235\\ 235\\ 235\\ 232\\ 235\\ 222\\ 572\\ 572\\ 572\\ 572\\ 572\\ 572\\ 57$		37	$1.34 \\ 1.25 \\ 1.27$
TABLE 1 Distribution of Primes 5 7 11 13 17 19 23 29 1426 810 327 374 152 184 121 62 1426 810 327 374 155 184 121 62 1426 810 327 374 155 184 377 213 134 2533 5534 3996 1556 1737 856 976 538 285 5534 3996 1556 1012 538 285 538 286 9547 5597 2156 2421 1203 1358 788 404 9547 5597 2156 2421 1203 1358 788 404 10879 6583 3440 1667 1464 2351 1356 693 11879 8742 1809 2157 1012 550 1356 693 12216 2783 3440 1667 2351 1356		31	$\begin{array}{c} 86\\172\\241\\316\\322\\552\\622\\776\\850\\927\\927\end{array}$		31	$1.68 \\1.66 \\1.64 \\1.64 \\1$
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		29	$\begin{array}{c} 62\\ 134\\ 134\\ 134\\ 235\\ 235\\ 235\\ 235\\ 235\\ 520\\ 642\\ 642\\ 642\\ 642\\ 642\\ 642\\ 642\\ 642$		29	
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		23	$\begin{array}{c} 121\\ 213\\ 206\\ 426\\ 538\\ 664\\ 788\\ 900\\ 1012\\ 1118\\ 1251\\ 1356\end{array}$		23	
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		19	$\begin{array}{c} 184\\ 377\\ 573\\ 573\\ 780\\ 976\\ 1175\\ 1358\\ 1566\\ 1755\\ 1948\\ 1255\\ 1948\\ 2251\\ 2351\end{array}$		19	
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		17	$\begin{array}{c} 152 \\ 306 \\ 497 \\ 677 \\ 677 \\ 856 \\ 1027 \\ 1203 \\ 1560 \\ 1560 \\ 1560 \\ 1506 \\ 1506 \\ 1667 \\ 1964 \end{array}$			
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		13	$\begin{array}{c} 374 \\ 714 \\ 714 \\ 714 \\ 737 \\ 737 \\ 737 \\ 737 \\ 737 \\ 737 \\ 737 \\ 742 \\ 742 \\ 740 \\ 767 \\ 111 \end{array}$			
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		1			1	5733 573
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		1	$\begin{array}{c} 32\\ 642\\ 944\\ 1555\\ 1556\\ 1238\\ 2787\\ 2336\\ 336$		11	$1.64 \\ 1.63 \\ 1.61 $
		2	$\begin{array}{c} 810\\ 1605\\ 2397\\ 2397\\ 3202\\ 3996\\ 3399\\ 5597\\ 7176\\ 7964\\ 8742\\ 9509\\ 9509\end{array}$		2	$2.18 \\ 2.17 \\ 2.16 $
$\begin{array}{c c} p_i \\ \hline p_i \hline$		5	$\begin{array}{c} 1426\\ 2830\\ 2830\\ 5534\\ 5534\\ 6846\\ 8197\\ 9547\\ 10879\\ 112217\\ 13585\\ 114879\\ 116226\\ 16226\end{array}$		5	$2.23 \\ 2.19 \\ 2.18 $
		N N	$\begin{array}{c} 5000\\ 10000\\ 15000\\ 25000\\ 35000\\ 35000\\ 85000\\ 60000\\ 55000\\ 6000\\ 600\\$		p_i	20000 40000 60000

182

MOHAN LAL

We studied R(n) and $J_r(n)$ for $n \leq 60000$, using a sieve method. In Table 1, we present the number of primes $n(p_i)$, for $5 \leq p_i \leq 59$ at steps of 5000. We note that for these 15 values of p_i , $n(p_i)/N$ is quite constant. Therefore, there is strong indication that the ratio $n(p_i)/N$ approaches a definite limit, as $N \to \infty$.

Further empirical study reveals that $n(p_i)$ is very roughly proportional to $N/(p_i \times \log p_i)$. This is shown in Table 2, where we tabulate

(3)
$$C(p_i) = p_i \times \log (p_i) \times n(p_i)/N$$

for N = 20000, 40000 and 60000. The values of $C(p_i)$ are quite consistent for these three values of N.

As far as R(n) is concerned, we note the empirical result that $R(n) \leq \lfloor \log(n) \rfloor + 3$, for $n \leq 60000$.

In conclusion, we might indicate, that though (3) describes the distribution $n(p_i)$ approximately, it is not entirely satisfactory, since it does not indicate why n(p+2) > n(p), in the case of twin primes, except for p = 5. One wonders if the expression (3) has any basis, heuristic or otherwise. The constants $C(p_i)$ do vary considerably for these fifteen values.

Memorial University of Newfoundland St. John's, Newfoundland, Canada