Iterates of a Number-Theoretic Function

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Abstract. Iterates of a function defined by the sum of the prime divisors of a number, where the multiple factors are counted multiply, are considered. The process of iteration is terminated at a prime. The density distribution of these primes is investigated empirically, for $N \leqq 60000$ and it is found to be quite constant. ■

Introduction. Let $n = \prod_i p_i^{\alpha_i}$ be the representation of n as a product of distinct primes and define a function,

$$J(n) = \sum_i \alpha_i p_i \,. \tag{1}$$

Thus the function $J(n)$ is defined to be the sum of the prime divisors of n and the multiple factors are counted multiply. From (1) it follows that $J(n)$ is completely additive, for $J(mn) = J(m) + J(n)$.

For n, a prime, $J(p) = p$ and for n, a composite number, $J(n) < n$. For $n = 4$, $J(4) = 4$ and this is considered to be an exceptional case and thus the number 4 behaves like a prime.

The rth iterate of $J(n)$ is defined by

$$J_r(n) = J(J_{r-1}(n)) \; ; \qquad J_1(n) = J(n) \,. \tag{2}$$

When n is a prime, each successive iterate gives rise to the same prime and we may say that the process of iteration converges. In what follows we will assign a value of 1 to r, for n = prime. For composite numbers, r takes definite positive integral values for the iteration (2) to converge. Thus for $n = 8$, $r = 3$ since $J_1(8) = 6$; $J_2(8) = 5$; $J_3(8) = 5$. In this manner, we shall associate with an integer n, a function $R(n)$ which defines the minimum number of iterates of $J(n)$ required to transform it into a prime. Thus, we define $R(n) = r$ and so, $R(30) = 3$, $R(24) = 4$, $R(10) = 2$.

Naturally the question arises, "What can be said about $R(n)$?"

Still, another interesting problem related to the iterates of $J(n)$ could be stated as follows. Suppose we apply the iterates of $J(n)$ to all integers less than or equal to N, we obtain a set of primes which are distributed all over the interval. Then, if $n(p_i)$ is the number of primes p_i between 1 and N inclusive, "Does the ratio $n(p_i)/N$ approach a definite limit, as N approaches infinity, for all p_i?" It should be stated that the prime $p = 3$ occurs only once and $n = 4$ is an exceptional case for which $R(4)$ is not defined in our context.

To find analytic solutions to these questions, which involve various partitions of a number into primes, would probably be difficult. In what follows we provide an empirical investigation of these problems.

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TABLE 1

Distribution of Primes

N \ p_i	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59
5000	1426	810	327	374	152	184	121	62	86	56	51	70	40	37	28
10000	2830	1605	649	714	306	377	213	134	172	104	88	113	88	76	50
15000	4188	2397	941	1049	497	573	306	183	241	156	139	170	139	97	77
20000	5534	3202	1241	1400	677	780	426	235	316	200	180	221	189	128	101
25000	6846	3996	1556	1737	856	976	538	285	392	235	221	289	223	156	121
30000	8197	4812	1860	2066	1027	1175	664	341	472	289	260	346	263	187	145
35000	9547	5597	2156	2421	1203	1358	788	404	552	332	287	398	303	221	171
40000	10879	6380	2477	2758	1360	1566	900	462	622	374	331	453	340	253	199
45000	12217	7176	2785	3086	1506	1755	1012	520	703	423	368	520	375	275	225
50000	13585	7964	3081	3440	1667	1948	1118	585	776	463	396	588	409	312	246
55000	14879	8742	3367	3767	1809	2157	1251	642	850	522	441	643	449	345	275
60000	16226	9509	3662	4111	1964	2351	1356	698	927	572	469	705	492	378	294

TABLE 2

Values of $C(p_i)$

N \ p_i	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59
20000	2.23	2.18	1.64	2.33	1.63	2.18	1.54	1.15	1.68	1.34	1.37	1.79	1.71	1.35	1.21
40000	2.19	2.17	1.63	2.30	1.64	2.19	1.62	1.13	1.66	1.25	1.26	1.83	1.54	1.33	1.20
60000	2.18	2.16	1.61	2.28	1.58	2.19	1.63	1.14	1.64	1.27	1.19	1.90	1.48	1.33	1.18

We studied $R(n)$ and $J_r(n)$ for $n \leqq 60000$, using a sieve method. In Table 1, we present the number of primes $n(p_i)$, for $5 \leqq p_i \leqq 59$ at steps of 5000. We note that for these 15 values of p_i, $n(p_i)/N$ is quite constant. Therefore, there is strong indication that the ratio $n(p_i)/N$ approaches a definite limit, as $N \to \infty$.

Further empirical study reveals that $n(p_i)$ is very roughly proportional to $N/(p_i \times \log p_i)$. This is shown in Table 2, where we tabulate

$$C(p_i) = p_i \times \log (p_i) \times n(p_i)/N \tag{3}$$

for $N = 20000$, 40000 and 60000. The values of $C(p_i)$ are quite consistent for these three values of N.

As far as $R(n)$ is concerned, we note the empirical result that $R(n) \leqq [\log(n)] + 3$, for $n \leqq 60000$.

In conclusion, we might indicate, that though (3) describes the distribution $n(p_i)$ approximately, it is not entirely satisfactory, since it does not indicate why $n(p + 2) > n(p)$, in the case of twin primes, except for $p = 5$. One wonders if the expression (3) has any basis, heuristic or otherwise. The constants $C(p_i)$ do vary considerably for these fifteen values.

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